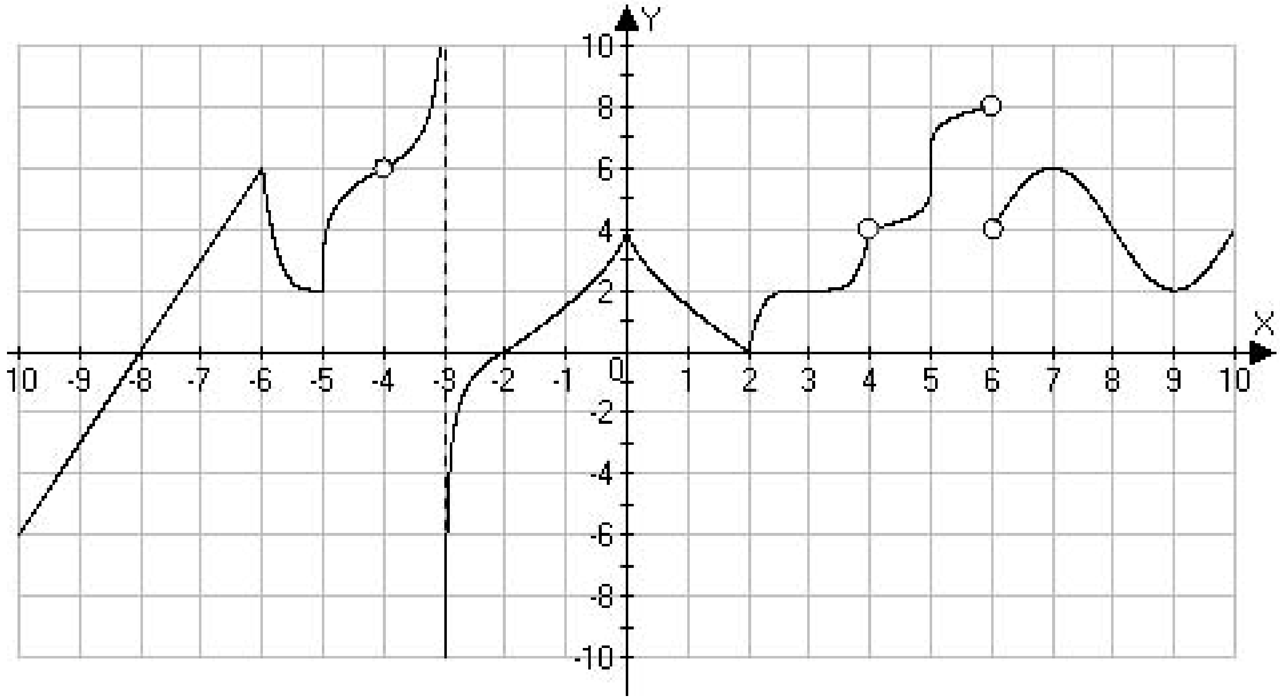
Math 1A – Final Exam – Fall 2010 Name____________________

Show your work for credit. Write all responses on separate paper. Do not use a calculator.

1. Consider the following graph for the function $y = f(x)$. Use the graph to answer the following questions as best as you can, based on the information in the graph.



a. What is $\lim_{x\to 6^-} f(x)$ if it exists, if not, explain why not.

b. Find $\lim_{x\to 6} f(x)$ or explain why it does not exist.

c. Is *f* continuous throughout its domain? Why or why not?

d. Where does *f* (*x*) is have a removable discontinuity? List all values of *x* where this is true.

e. Find all values of *a* so that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = 0$

f. Find all values of *a* so that $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = \infty$

g. What is $f'(-8)$?

h. Based on the graph, what is the domain of $y = f'(x)$? Use interval notation.

i. Where on the graph does it appear that $\lim_{h\to 0^-} \frac{f(x)-f(x+h)}{h} = -\lim_{h\to 0^+} \frac{f(x)-f(x+h)}{h} = 2$?

2. Find the limit if it exists, or state why it does not exist.

a. $\lim_{x\to 3}\dfrac{x^3-27}{x^3-3x^2+x-3}$ b. $\lim_{t\to 2}\dfrac{2-t}{\left|4-t^2\right|}$ c. $\lim_{\theta\to 2}(\theta-2)^{\sin(\pi\theta)}$

3. Simplify the derivative *dy*/*dx* for each:

a. $y=\dfrac{(x+\lambda)^5}{x^5+\lambda^5}$ b. $y=\dfrac{\arctan(mx)}{x}$ c. $y=\ln\left|\dfrac{x^2-1}{3x+2}\right|$

4. Consider the function $f(x)=\sqrt{1-\tan(4x)}$

a. Find an equation for the line tangent to $y=f(x)$ at (0,1).

b. Use the linearization at (0,1) to approximate $f\left(\dfrac{\pi}{12}\right)$.

5. Find values of a and b so that the parabola $y = ax^2 + bx + c$ passes through the point (–1,5) and has tangent lines at $x = –5$ and $x = –3$ have slopes 2 and 6, respectively.

6. The angle of elevation of the sun is increasing at a rate of 0.25 radians per hour. How fast is the shadow cast by a 100 meters tall building decreasing when the angle of elevation of the sun is $\pi/3$?

7. Sketch a graph of a function that satisfies the given conditions:
f is odd, $f'(x)>0$ for $x<-2$, $f'(x)<0$ for $-2<x<0$,
$f''(x)>0$ on $(-\infty,-3)\cup(-1,0)$, $f''(x)<0$ on $(-3,-1)$ and $\lim_{x\to\infty}f(x)=1$.

8. Let $f(x)=\cos^2 x-2\sin x$ on the interval $[0, 2\pi]$.

a. Find all critical numbers for $f(x)$.

b. Find all inflection points for $f(x)$.

c. Sketch a graph for $y = f(x)$ showing the key features.

9. Consider $f(x)=x^{1/4}$ on the interval [16,17]

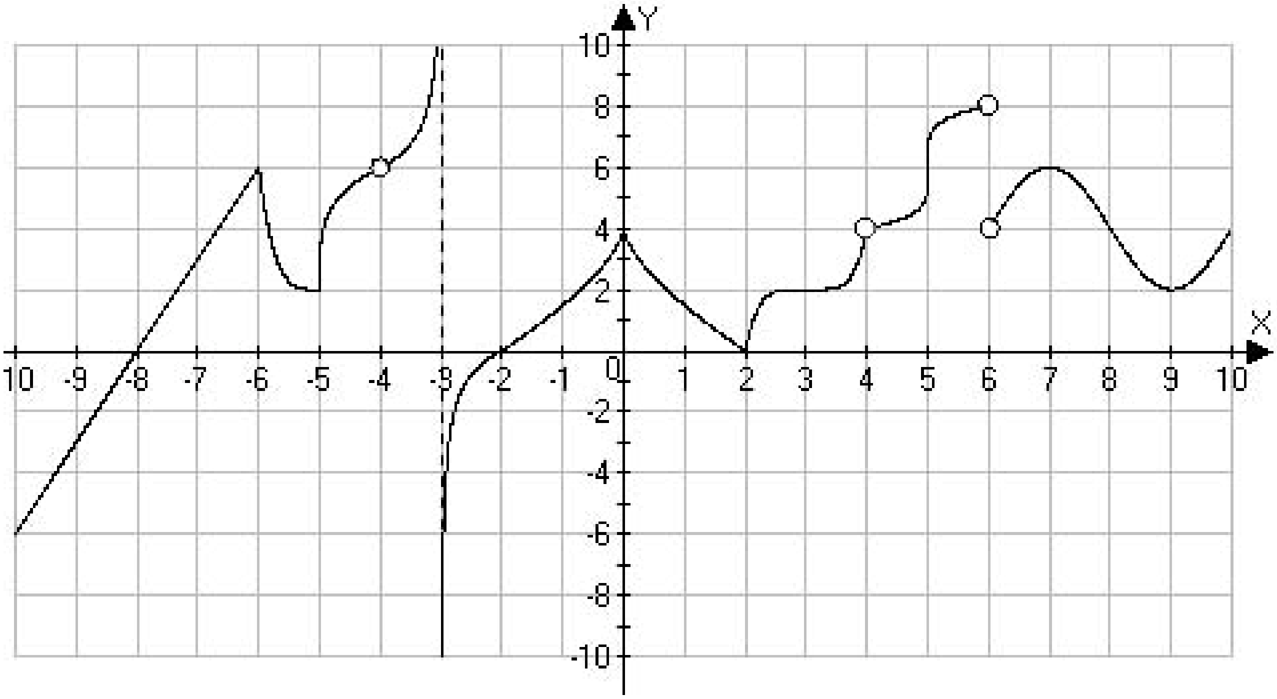
a. State the conditions of the mean value theorem and explain why f meets the conditions on this interval.

b. Find a value of c whose existence is guaranteed by the mean value theorem.

10. Find the dimensions of the largest rectangle that will fit above the x-axis and below $y=\dfrac{1}{1+x^2}$

Math 1A – Final Exam Solutions – Fall 2010

1. Consider the following graph for the function $y = f(x)$. Use the graph to answer the following questions as best as you can, based on the information in the graph.



a. What is $\lim_{x \to 6^-} f(x)$ if it exists, if not, explain why not.

SOLN: $\lim_{x \to 6^-} f(x) = 8$

b. Find $\lim_{x \to 6} f(x)$ or explain why it does not exist.

SOLN: $\lim_{x \to 6} f(x)$ does not exist since $\lim_{x \to 6^+} f(x) = 4 \neq 8 = \lim_{x \to 6^-} f(x)$

c. Is *f* continuous throughout its domain? Why or why not?
SOLN: Yes, *f* is undefined at -4, -3, 4 and 6 where the discontinuities are.

d. Where does $f(x)$ is have a removable discontinuity? List all values of *x* where this is true.
SOLN: There are removable discontinuities at -4 and 4.

e. Find all values of *a* so that $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0$

SOLN at $x = 3$, 7 and 9

f. Find all values of *a* so that $\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \infty$

SOLN: Looks like the slopes approach positive infinity from the right at $x = -5$ and 5.

g. What is $f'(-8)$?

SOLN: 3

h. Based on the graph, what is the domain of $y = f'(x)$? Use interval notation.

SOLN: (-10,-6) , (-6,-5) , (-5,-4), (-4,-3) , (-3,0), (0,2), (2, 4), (4,5), (5,6), and (6,10)

i. Where on the graph does it appear that $\lim_{h \to 0^-} \frac{f(x) - f(x+h)}{h} = -\lim_{h \to 0^+} \frac{f(x) - f(x+h)}{h} = 2$?

SOLN: At $x = 0$.

2. Find the limit if it exists, or state why it does not exist.

a. $\lim_{x\to 3}\frac{x^3-27}{x^3-3x^2+x-3}=\lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{x^2(x-3)+(x-3)}=\lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{(x-3)(x^2+1)}=\lim_{x\to 3}\frac{x^2+3x+9}{x^2+1}=\frac{27}{10}$

b. $\lim_{t\to 2^+}\frac{2-t}{|4-t^2|}=\lim_{t\to 2^+}\frac{2-t}{t^2-4}=\lim_{t\to 2^+}\frac{2-t}{(t-2)(t+2)}=\lim_{t\to 2^+}\frac{-1}{t+2}=-\frac{1}{4}$ which is not equal to the limit from the other side: $\lim_{t\to 2^-}\frac{2-t}{|4-t^2|}=\lim_{t\to 2^-}\frac{2-t}{4-t^2}=\lim_{t\to 2^+}\frac{2-t}{(2-t)(2+t)}=\lim_{t\to 2^+}\frac{1}{t+2}=\frac{1}{4}$, so the limit does not exist.

c. $\lim_{\theta\to 2^+}(\theta-2)^{\sin(\pi\theta)}$ For this limit, use a logarithm. Let $y=(\theta-2)^{\sin(\pi\theta)}$ so that $\ln y=\sin(\pi\theta)\ln(\theta-2)=\frac{\ln(\theta-2)}{\csc(\pi\theta)}$ and $\lim_{\theta\to 2^+}\ln y=\lim_{\theta\to 2^+}\frac{\ln(\theta-2)}{\csc(\pi\theta)}$ is an ∞/∞ situation where we can apply L'Hospital's rule: $\lim_{\theta\to 2^+}\frac{\ln(\theta-2)}{\csc(\pi\theta)}=\lim_{\theta\to 2^+}\frac{-1}{(\theta-2)\csc(\pi\theta)\cot(\pi\theta)}=\lim_{\theta\to 2^+}\frac{-\sin^2(\pi\theta)}{\pi(\theta-2)\cos(\pi\theta)}$. This last limit is a 0/0 situation, so we apply L'Hospital's rule again:

$$\lim_{\theta\to 2^+}\frac{-\sin^2(\pi\theta)}{\pi(\theta-2)\cos(\pi\theta)}=\lim_{\theta\to 2^+}\frac{-\pi\sin(\pi\theta)\cos(\pi\theta)}{\pi\cos(\pi\theta)-\pi^2(\theta-2)\sin(\pi\theta)}=\frac{0}{\pi}=0 \text{ so } \lim_{\theta\to 2^+}\ln y=0\Rightarrow \boxed{\lim_{\theta\to 2^+}y=1}$$

3. Simplify the derivative dy/dx for each:

a.
$$y=\frac{(x+\lambda)^5}{x^5+\lambda^5}\Rightarrow\frac{dy}{dx}=\frac{5(x^5+\lambda^5)(x+\lambda)^4-5x^4(x+\lambda)^5}{(x^5+\lambda^5)^2}=\frac{5(x+\lambda)^4\left[(x^5+\lambda^5)-x^4(x+\lambda)\right]}{(x^5+\lambda^5)^2}$$

$$=\frac{5\lambda(x+\lambda)^4(\lambda^4-x^4)}{(x^5+\lambda^5)^2}=\frac{5\lambda(x+\lambda)^4(\lambda^4-x^4)}{(x+\lambda)^2(x^4-\lambda x^3+\lambda^2x^2-\lambda^3x+\lambda^4)^2}=\boxed{\frac{5\lambda(x+\lambda)^2(\lambda-x)(\lambda^3+\lambda^2x+\lambda x^2-x^3)}{(x^4-\lambda x^3+\lambda^2x^2-\lambda^3x+\lambda^4)^2}}$$

b. $$y=\frac{\arctan(mx)}{x}\Rightarrow\frac{dy}{dx}=\frac{\frac{mx}{1+m^2x^2}-\arctan(mx)}{x^2}\left(\frac{1+m^2x^2}{1+m^2x^2}\right)=\boxed{\frac{mx-(1+m^2x^2)\arctan(mx)}{x^2(1+m^2x^2)}}$$

c. $$y=\ln\left|\frac{x^2-1}{3x+2}\right|=\frac{3x+2}{x^2-1}\cdot\frac{2x(3x+2)-3(x^2-1)}{(3x+2)^2}=\boxed{\frac{3x^2+4x+3}{(3x+2)(x^2-1)}}$$

4. Consider the function $f(x)=\sqrt{1-\tan(4x)}$

a. Find an equation for the line tangent to $y=f(x)$ at (0,1).

SOLN: $f'(x)=\frac{-2\sec^2(4x)}{\sqrt{1-\tan(4x)}}\Rightarrow f'(0)=-2$, so an equation of the tangent line is $y-1=-2x$

b. Use the linearization at (0,1) to approximate $f\left(\frac{\pi}{12}\right)$.

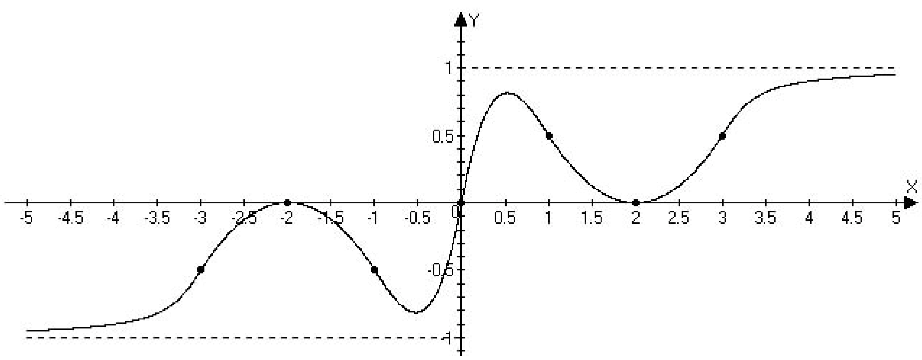
$$f\left(\frac{\pi}{12}\right)\approx f(0)+f'(0)\frac{\pi}{12}=1-\frac{\pi}{6}$$

5. Find values of a, b and c so that the parabola $y = ax^2 + bx + c$ passes through the point $(-1,5)$ and has tangent lines at $x = -5$ and $x = -3$ have slopes 2 and 6, respectively.

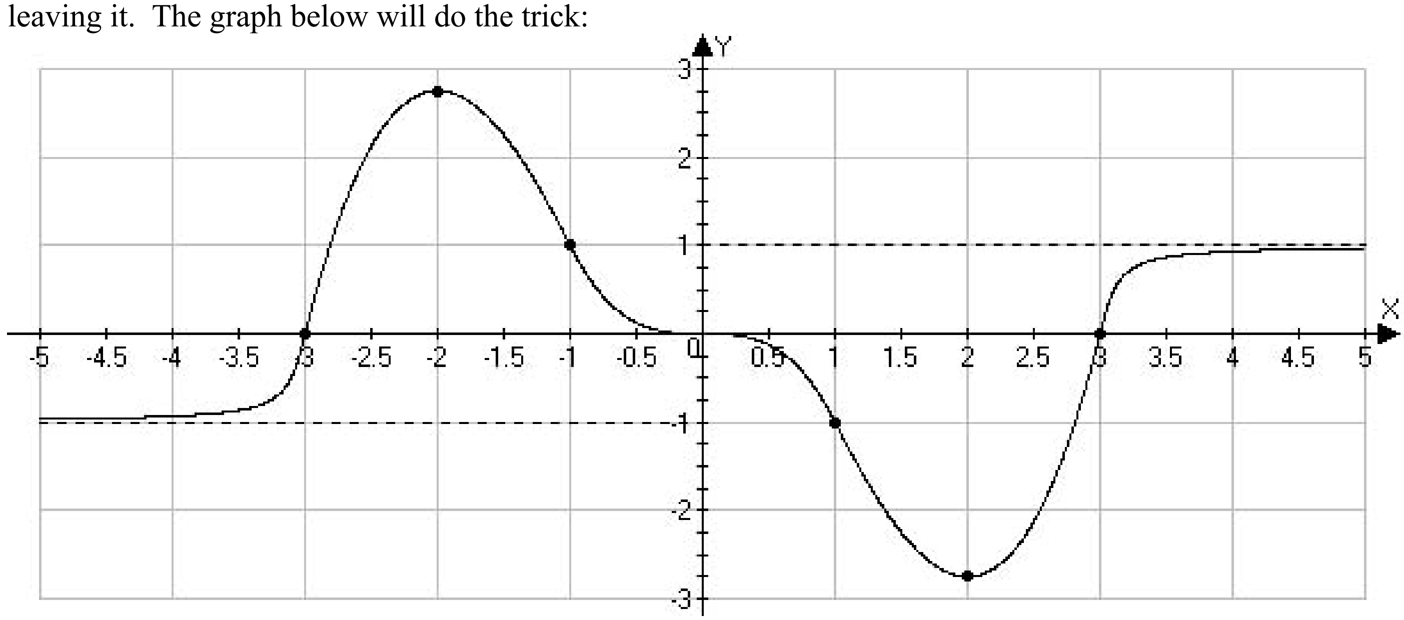
SOLN: The three constraints lead to a linear 3X3 system:
$$\begin{aligned} a-b+c&=5 \\ -10a+b&=2 \\ -6a+b&=6 \end{aligned}$$
The difference of the last two equations indicate that $4a = 4$ so $a = 1$ and thus $b = 12$. Plugging these into the first equation yields $c = 16$ and so the parabola is $y = x^2 + 12x + 16$ Hey, this test was given on December 16th!

6. The angle of elevation of the sun is increasing at a rate of 0.25 radians per hour. How fast is the shadow cast by a 100 meters tall building decreasing when the angle of elevation of the sun is $\pi/3$?
SOLN: Let x = the length of the shadow and let θ = the angle of elevation of the sun. Then $\tan\theta = 100/x$ and so $\sec^2\theta\,(d\theta/dt) = (-100/x^2)(dx/dt)$. Now when $\theta = \pi/3$, $x = 100/\sqrt{3}$ and $\sec^2(\pi/3) = 4$. Plugging in all these values, we get $4(0.25) = (-300/1000)(dx/dt)$ so that $dx/dt = -10/3$ meters per second is the rate of change of the shadow's length.

7. Sketch a graph of a function that satisfies the given conditions:
f is odd, $f'(x) > 0$ for $x < -2$, $f'(x) < 0$ for $-2 < x < 0$,
$f''(x) > 0$ on $(-\infty,-3)\cup(-1,0)$, $f''(x) < 0$ on $(-3,-1)$ and $\lim_{x\to\infty} f(x) = 1$.



Oh, crap – that doesn't fit $f'(x) < 0$ for $-2 < x < 0$. And it took a really long time to work up, so I'm leaving it. The graph below will do the trick:



8. Let $f(x)=\cos^2 x-2\sin x$ on the interval [0, 2π].

   a. Find all critical numbers for $f(x)$.

   SOLN: $f'(x)=-2\cos x\sin x-2\cos x=-2\cos x(\sin x+1)=0 \Leftarrow \boxed{x=\frac{\pi}{2},\frac{3\pi}{2}}$

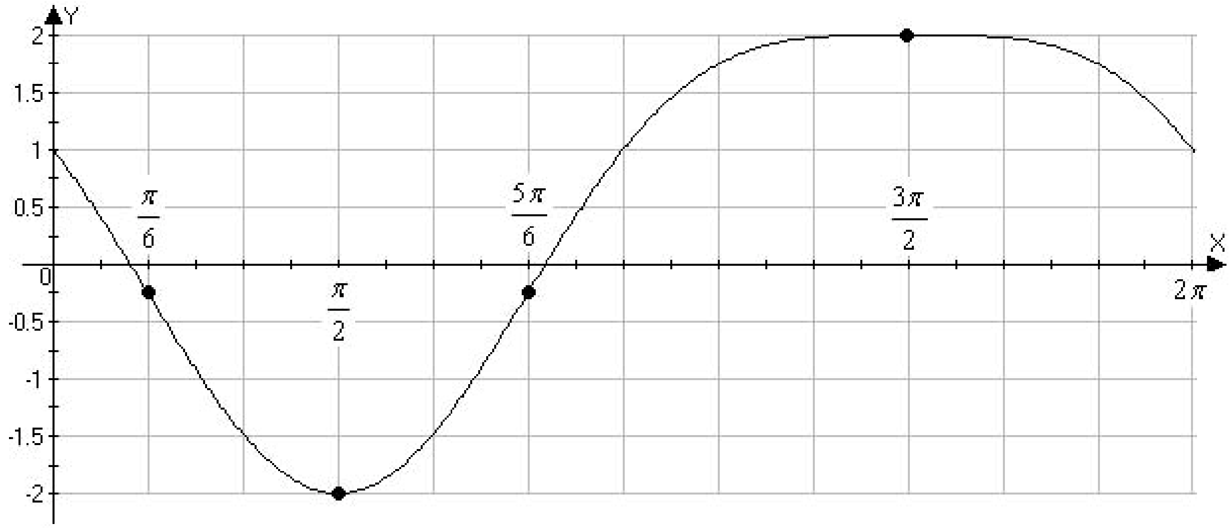
   b. Find all inflection points for $f(x)$.

   SOLN: $f''(x)=2(2\sin^2 x-1)+2\sin x=2\sin^2 x+\sin x-1=(2\sin x-1)(\sin x+1)=0$

   where $x=\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2}$, but doesn't change sign at $\frac{3\pi}{2}$, so the inflection points are $\left(\frac{\pi}{6},-\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6},-\frac{1}{4}\right)$.

   c. Sketch a graph for $y=f(x)$ showing the key features.

   SOLN:



9. Consider $f(x)=x^{1/4}$ on the interval [16,17]

   a. State the conditions of the mean value theorem and explain why *f* meets the conditions on this interval.

   SOLN: The conditions of the MVT are that *f* is continuous on [16,17] and differentiable on (16,17).

   b. Find a value of *c* whose existence is guaranteed by the mean value theorem.

   SOLN: $f'(c)=\frac{1}{4c^{3/4}}=\frac{f(17)-f(16)}{17-16}=\sqrt[4]{17}-2 \Leftrightarrow c=\left(\frac{1}{4\left(\sqrt[4]{17}-2\right)}\right)^{4/3} \approx 16.49557975$

10. Find the dimensions of the largest rectangle that will fit above the *x*-axis and below $y=\frac{1}{1+x^2}$

   SOLN: Since the function has *y*-axis symmetry, the rectangle will have the same symmetry and the area will be A = height*width = $A(x)=\frac{2x}{1+x^2} \Rightarrow A'(x)=\frac{2(1+x^2)-4x^2}{(1+x^2)^2}=0 \Leftrightarrow x^2=1$ so the rectangle will be 2 units wide and ½ unit tall, for an area of 1 square unit.

Rejects:

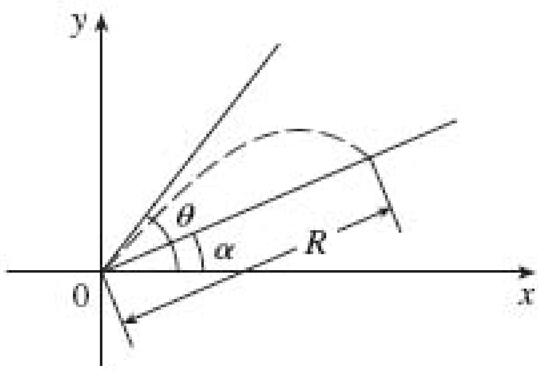
1. If a projectile is fired with an initial velocity v at an angle of inclination θ from the horizontal, then its trajectory, neglecting air resistance, is the parabola

$$y = (\tan\theta)x - \frac{g}{2v^2\cos^2\theta}x^2 \qquad 0 < \theta < \frac{\pi}{2}$$

   a. Suppose the parabola is fired from the base of a plane that is inclined at an angle α, $\alpha > 0$, from the horizontal, as shown in the figure. Show that the range of the projectile, measure up the slope, is given by

$$R(\theta) = \frac{2v^2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}$$

   b. Suppose the parabola is fired from the base of a plane that is inclined at an angle α below the horizontal (as shown in the diagram below.) Determine the range *R* in this case, and determine the angle at which the projectile should be fired to maximize *R*.



2. Suppose a function $y = f(x)$ satisfies the equation $y^2\cos(\pi x) + x^2y + y^3 = 1$ in a neighborhood of the point (1,1). Find an equation for the tangent line at (1,1).

3. Use a linear approximation to estimate $\arctan\left(\frac{3}{4}\right)$ by considering the tangent line to $y = \arctan x$ at $x = 1$. Approximate to 3 significant digits.

4. A spherical balloon is filling with water at a rate of π cm^3/sec. How fast is the surface area increasing when the radius is 2 cm? Useful formulas might include $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$.

5. Show there are no values of x in the interval (–1,1) that satisfy the conclusion of the mean value theorem for $f(x) = \frac{1}{x}$. Why does this not contradict the theorem?

6. If a resistor of r ohms is connected across a battery of V volts with internal resistance R ohms, then the power in watts in the external resistor is $P(r) = \frac{V^2 r}{(r+R)^2}$. If V and R are constant by r varies, what is the maximum value of the power?

7. Consider the equation $\sin 3x = 1 - x^3$

a. Use the intermediate value theorem to prove that this equation has a solution in $\left[0, \frac{\pi}{3}\right]$

b. Use Newton's method to find the next estimate to the solution starting from $x_1 = \frac{\pi}{6}$.

8. Find the antiderivative of $f(x) = \frac{8}{x^2}$ which has $y = x$ as a tangent line.